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SERIES
Textbook

For Students Doing a Course of Higher Mathematics in English

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Составители:
Доц. ***В. Н. Гавдзинский***, ст. преп. ***Л. Н. Коробова***

Методическое пособие содержит разделы высшей математики “Числовые ряды”, “Степенные ряды” и “Ряды Фурье” на английском языке и предназначено для студентов академии, изучающих математику на английском языке.

Основные теоремы и формулы приведены с доказательством, а также даны решения типовых задач и задания для самостоятельной работы.

CONTENTS

S E R I E S

I. NUMERICAL SERIES

§ 1.1. Definition of the sum of a series

Consider an infinite sequence of numbers $a_1, a_2, \dots a_n, \dots$

Definition. An expression

$$a_1 + a_2 + ... + a_n + ...$$

is called a series, and the numbers $a_1, a_2, \dots a_n, \dots$ are called the terms of the series.

A series is briefly written as

$$\sum_{n=1}^{n=\infty} a_n ,$$

and a_n is called the *n*th term or the general term of the series.

The sequence $\{S_n\}$defined by

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + ... + a_n = \sum_{k=1}^{k=n} a_k$$

is the **sequence of partial sums** of the series, S_n being the ***n*th partial sum**.

Definition. If the sequence of partial sums of the given series has a definite limit as $n \to \infty$, i.e.

$$\lim_{n\to\infty} S_n = S < \infty$$

the series is said to be **convergent** and the number *S* is called the sum of the series.

In this case we write

$$S = a_1 + a_2 + ... + a_n + \dots$$

If the sequence S_n does not tend to any finite limit, the series is said to be **divergent**.

§ 1.2. Geometrical Series

Geometrical series are series of the form

$$a + aq + aq^2 + ... + aq^{n-1} + ... = \sum_{n=1}^{n=\infty} aq^{n-1} \qquad (1.2.1)$$

in which *a* and *q* are fixed real numbers and $a \neq 0$.

The sum of the first *n* terms of the series (1.2.1) is the sum of *n* terms of the corresponding geometrical progression

$$S_n = \frac{a - aq^n}{1-q} \qquad (1.2.2)$$

If $|q|<1$ then $\lim\limits_{n\to\infty} S_n = a\lim\limits_{n\to\infty}\frac{1-q^n}{1-q} = \frac{a}{1-q}$.

Consequently, a geometric series with $|q|<1$ is convergent and its sum is

$$S = \frac{a}{1-q}. \tag{1.2.3}$$

If $|q|>1$ then $\lim\limits_{n\to\infty} q^n = \infty$, and hence $\lim\limits_{n\to\infty} S_n = \infty$, that is the series (1.2.1) is divergent.

Now, let $q = 1$. The series $a + a + ... + a + ... (a \neq 0)$ has the nth partial sum $S_n = na$, which tends to infinity together with n: $\lim\limits_{n\to\infty} S_n = \infty$.

If $q = -1$ we get the series $a - a + a - a + ...$. Its partial sums take on values $S_1 = a$, $S_2 = 0$, $S_3 = a$, $S_4 = 0$, etc., that is S_n does not tend to any limit.

Thus, a geometric series is convergent for $|q|<1$ and divergent for $|q|\geq 1$.

Example. You drop a ball from meter above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance qh, where q is positive but less than 1. Find the total distance the ball travels up and down.

Solution. The total distance is

$$S = a + 2aq + 2aq^2 + 2aq^3 + ... = a + \frac{2aq}{1-q} = a\frac{1+q}{1-q}.$$

If a = 6m and $q = \frac{2}{3}$, for instance, the distance is

$$S = 6\frac{1+(2/3)}{1-(2/3)} = 30\text{m}.$$

§ 1.3. Basic Properties of Series

Convergent series have certain properties that allow operations to be performed on them, as on finite sums.

I. If the series

$$a_1 + a_2 + ... + a_n + ... \tag{1.3.1}$$

has the sum S, the series

$$aa_1 + aa_2 + ... + aa_n + ..., \tag{1.3.2}$$

obtained by multiplying all the terms of the first series by the same factor a, has the sum aS, because the sum σ_n of the first n terms of (1.3.2) is

$$\sigma_n = aa_1 + aa_2 + ... + aa_n = aS_n,$$

and hence $\lim\limits_{n\to\infty}\sigma_n = \lim\limits_{n\to\infty} aS_n = aS$.

II. Convergent series can be added and subtracted by terms, i.e. if

$$a_1 + a_2 + ... + a_n + ... = A,$$
$$b_1 + b_2 + ... + b_n + ... = B,$$

the series $(a_1 \pm b_1)+(a_2 \pm b_2)+...+(a_n \pm b_n)+...$
is also convergent, and its sum is $A \pm B$.

III. The convergence and divergence of a series is unaltered by removing or adding a finite number of terms at the beginning.
You must prove the last two properties by yourself.

§ 1.4. A necessary (but not sufficient) condition for convergence

Theorem. If a series is convergent its *n*th term tends to zero as $n \to \infty$.
Proof.
We have $S_n = a_1 + a_2 + ... + a_{n-1} + a_n = S_{n-1} + a_n$.
If the series is convergent we obtain

$$\lim_{n\to\infty} S_{n-1} = S \text{ and } \lim_{n\to\infty} S_n = S.$$

Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1},$$

we get

$$\lim_{n\to\infty} a_n = S - S = 0.$$

It should be pointed out that if the *n*th term of a series tends to zero as $n \to \infty$ that this fact is not sufficient for the convergence of the series.
To do this we consider the following series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} + ...$$

whose *n*th term tends to zero. Here we have

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + ... + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

Consequently

$$\lim_{n\to\infty} S_n = \infty$$

and the given series diverges.
Keeping this in mind we can state the *n*th – term test for divergence:

If $\lim\limits_{n\to\infty} a_n \neq 0$, or if $\lim\limits_{n\to\infty} a_n$ fails to exist, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Example. 1.The series $\sum\limits_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty \neq 0$ as $n \to \infty$.

2. The series $\sum\limits_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$ as $n \to \infty$.

3. The series $\sum\limits_{n=1}^{\infty} (-1)^n$ diverges because $\lim\limits_{n\to\infty} (-1)^n$ does not exist.

§ 1.5. Series with positive terms. Comparison test

Let us consider a positive series whose all terms are positive:

$$a_1 + a_2 + \ldots + a_n + \ldots \quad (a_n) > 0 \tag{1.5.1}$$

Lemma. If the partial sums of a positive series are bounded above, that is

$$S_n \le M, \quad M = \text{const}$$

the series is convergent.

Proof.

Since all the terms of the series are positive, its *n*th partial sum increases on infinite increase of *n*:

$$S_1 < S_2 < \ldots < S_n < \ldots .$$

But if a monotone increasing sequence is bounded above by a constant number it has a limit which does not exceed that number either. Hence

$$\lim_{n\to\infty} S_n = S \le M.$$

Conversely, if a positive series is convergent its partial sums are less than the sum of the series: $S_n < S$.

Comparison Test. Consider two positive series

$$\sum_{n=1}^{\infty} a_n \tag{1.5.2}$$

and

$$\sum_{n=1}^{\infty} b_n \tag{1.5.3}$$

satisfying the condition that each term of the series (1.5.2) does not exceed the corresponding term of the series (1.5.3), i.e.

$$a_n \le b_n \quad (n = 1,2,\ldots) \tag{1.5.4}$$

Then:

1. If the series (1.5.3) is convergent the series (1.5.2) is also convergent.
2. If the series (1.5.2) is divergent the series (1.5.3) is also divergent.

Proof. We start with the first part of the statement.

Let us set

$$A_n = \sum_{k=1}^{k=n} a_k , \qquad B_n = \sum_{k=1}^{k=n} b_k .$$

By the condition, series (1.5.3) is convergent and therefore $B_n < B$, where B is the sum of the second series.

It follows from the inequality (1.5.4) that

$$A_n \le B_n < B$$

which means that the partial sums of series (1.5.2) are bounded above. But then the lemma implies that series (1.5.2) is convergent.

The second part of this theorem can be proved in the same way.

Example. We can establish the convergence of the series

$$5+\frac{2}{3}+1+\frac{1}{7}+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{n!}+...$$

by ignoring the first four terms and comparing the remainder with the convergent geometric series

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2!}+\frac{1}{4}+\frac{1}{8}+... \; .$$

§ 1.6. The Integral Test

Let $a_n = f(n)$ where $f(x)$ is a continuous, positive, decreasing function of x for all $x \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ both converge or both diverge.

Proof. We start with the assumption that $f(x)$ is a decreasing function with $f(n) = a_n$ for every n. This leads us to observation that the rectangles in *Fig..1*, which have areas a_1, $a_2, ...a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x=1$ to $x = n+1$. That is $\int_1^{n+1} f(x)dx \leq a_1 + a_2 + ... + a_n$

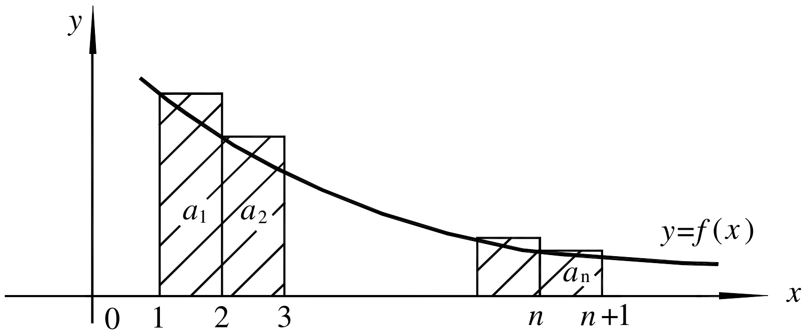

Fig.1

In *Fig.2.* the rectangles have been faced to the left instead of to the right.

If we momentarily disregard the first rectangle, of area a_1, we see that

$$a_2 + a_3 + ... + a_n \leq \int_1^n f(x)dx$$

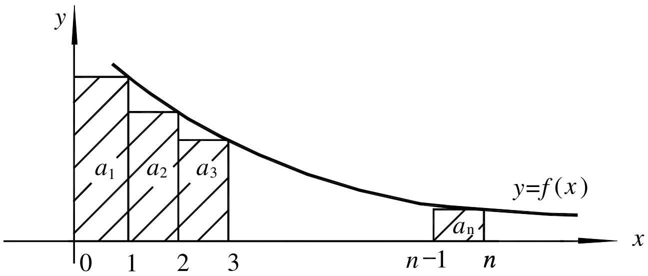

Fig.2

If we include a_1 we have

$$a_1 + a_2 + \ldots + a_n \le a_1 + \int_1^n f(x)dx$$

Combining these results gives

$$\int_1^{n+1} f(x)dx \le a_1 + a_2 + \ldots + a_n \le a_1 + \int_1^n f(x)dx .$$

If the integral $\int_1^{+\infty} f(x)dx$ is finite, the right-hand inequality shows that $\sum_{n=1}^{\infty} a_n$ is also finite. But if $\int_1^{+\infty} f(x)dx$ is infinite, the left-hand inequality shows that the series is also infinite.

Hence the series and the corresponding improper integral are both convergent or both divergent.

Example. **The p – series**.

If p is a real constant, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

converges if $p > 1$ and diverges if $p \le 1$.To prove this, let $f(x) = \dfrac{1}{x^p}$. Then we have

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^p}dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^{\infty} = \frac{1}{1-p} \lim_{x\to\infty}\left(x^{-p+1} - 1\right).$$

So integral is convergent if $p > 1$ because in this case $\lim_{x\to\infty} x^{-p+1} = 0$ and

$$\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{p-1}.$$

If $p < 1$ the integral is divergent because in the latter case we have $\lim_{x\to\infty} x^{-p+1} = \infty.$

For $p = 1$ the integral is also divergent: $\int_1^{\infty} \frac{1}{x}dx = \ln x\Big|_1^{\infty} = \infty.$

Therefore the series is convergent for $p > 1$ and divergent for $p \le 1.$

For $p = 1$ we have the **harmonic** series :

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

The divergence of this series was devised by the French theologian, mathematician, physicist and bishop Nicole Oresme (1320 – 1382), (pronounce "or-rem") by grouping the terms of the series in some special way.

§ 1.7. The Limit Comparison Test

If the limit of the ratio of a_n to b_n exists and is not equal to zero, i.e.

$$\lim_{n\to\infty}\frac{a_n}{b_n}=A>0$$

then series (1.5.2) and (1.5.3) are both convergent or both divergent.

Proof. From the definition of a limit it follows that, given any $\varepsilon>0$, there is a number N such that for all $n>N$ the inequality $\left|\frac{a_n}{b_n}-A\right|<\varepsilon$ holds, or equivalently

$$A-\varepsilon<\frac{a_n}{b_n}<A+\varepsilon,$$

where ε is assumed to be so small that $A-\varepsilon>0$. Let us suppose that series (1.5.3) is convergent, then the series with the general term $(A+\varepsilon)b_n$ is convergent, and by the comparison test, series (1.5.2) is also convergent because $a_n<(A+\varepsilon)b_n$ for all $n>N$.

Conversely, if series (1.5.3) is divergent, the inequality $a_n>(A-\varepsilon)b_n$ implies that series (1.5.2) is divergent as well.

The theorem has been proved.

The difficulty in applying the comparison tests is that for the given series it is necessary to construct another series with which the former can be compared. The harmonic series is often taken as a "standard" divergent series, whereas as a "standard" convergent series we usually take a convergent geometric series or a convergent *p*-series.

Example. Which of the following series converge and which diverge?

1) $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\ldots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$;

2) $\frac{101}{3}+\frac{102}{10}+\frac{103}{29}+\ldots=\sum_{n=1}^{\infty}\frac{100+n}{n^3+2}$;

3) $\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2^n-1}$.

Solution.

1) Let $a_n=\frac{2n+1}{(n+1)^2}$ and we take $b_n=\frac{1}{n}$.

Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 \neq 0$, so

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ diverges.

2) Let $a_n = \frac{100+n}{n^3+2}$. When n is large, this ought to compare with $\frac{n}{n^3} = \frac{1}{n^2}$. So we take $b_n = \frac{1}{n^2}$ and apply the limit comparison test:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges and } \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+100n^2}{n^3+1} = 1 \neq 0.$$

So $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{100+n}{n^3-2}$ converges.

3) Let $a_n = \frac{1}{2^n-1}$ and we take $b_n = \frac{1}{2^n}$. (We know that 2^{n-1} behaves somewhat like 2^n when n is large.) Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{1}{2}\right)^n} = 1 \neq 0.$$

Because $\sum_{n=1}^{\infty} b_n$ converges, we conclude, that $\sum_{n=1}^{\infty} a_n$ converges too.

§ 1.8. D'Alembert's Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = D$.

Then:

1. The series converges if $D<1$.

2. The series diverges if $D>1$.

3. The series may converge or it may diverge if $D=1$. (The test provides no information.)

Proof.

1. $D<1$. By the definition of a limit, it is always possible to choose a number N such that for all $n \geq N$ there holds the inequality

$$\frac{a_{n+1}}{a_n} < D + \varepsilon = \rho$$

where ε is chosen sufficiently small so that $\rho < 1$. Then

$$a_{N+1} < \rho \cdot a_N$$
$$a_{N+2} < \rho^2 \cdot a_N$$
$$a_{N+3} < \rho^3 \cdot a_N$$
$$\dots\dots\dots\dots\dots\dots$$

It follows that the terms of the series

$$a_{N+1} + a_{N+2} + a_{N+3} + \dots \tag{1.8.1}$$

are smaller than the corresponding terms of the convergent geometric series

$$\rho \cdot a_N + \rho^2 \cdot a_N + \rho^3 \cdot a_N + \dots \tag{1.8.2}$$

By comparison test series (1.8.1) is also convergent and the given series $\sum_{n=1}^{\infty} a_n$ is also convergent since the series $\sum_{n=N+1}^{\infty} a_n$ is convergent.

2. Now let $D > 1$. Then there is a number N such that for all $n \ge N$ the inequalities

$$\frac{a_{n+1}}{a_n} > 1, \text{ i.e. } a_{n+1} > a_n$$

hold. This means that every term of the series is greater than the preceding one and, since they are all positive, the necessary condition for convergence is violated. Hence, the series is divergent.

3. $D = 1$

The two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ show that some other test for convergence must be used when $D = 1$.

For $\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$.

In both cases $D = 1$, yet the first series diverges while the second one converges.

Example. Use D'Alembert's test to investigate the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)}$ **b)** $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

Solution.

a) If $a_n = \dfrac{(n!)^2}{(2n)!}$, then $a_{n+1} = \dfrac{((n+1)!)^2}{(2n+2)!}$, and

$$D = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n!)^2(n+1)^2(2n)!}{(2n)!(2n+1)(2n+2)(n!)^2} = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}.$$

The series is convergent because $D = \dfrac{1}{4}$ is less than 1.

b) If $a_n = \dfrac{n!}{5^n}$, then $a_{n+1} = \dfrac{(n+1)!}{5^{n+1}}$, and

$$D = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!\cdot 5^n}{5^{n+1}(n)!} = \frac{1}{5}\lim_{n\to\infty}(n+1) = \infty.$$

The series diverges because $D > 1$.

§ 1.9. The *n*th – Root Test. (Cauchy's Root Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and suppose that $\lim_{n\to\infty}\sqrt[n]{a_n} = C.$

Then:

1. The series converges if $C < 1$,

2. the series diverges if $C > 1$,

3. the test is not conclusive if $C = 1$.

Proof.

1. $C < 1$. Choose an $\varepsilon > 0$ so small that $C + \varepsilon < 1$ also. Then there exists an index N such that

$\sqrt[n]{a_n} < C + \varepsilon$, when $n \ge N$.

It is also true that

$a_n < (C + \varepsilon)^n$ for $n \ge N$.

Now, $\sum_{n=N}^{\infty}(C+\varepsilon)^n$, a geometric series with ratio $C + \varepsilon < 1$, converges. By comparison test, $\sum_{n=N}^{\infty} a_n$ converges, from which it follows that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_{N-1} + \sum_{n=N}^{\infty} a_n \text{ converges.}$$

2. $C > 1$. For all indices beyond some integer N, we have $\sqrt[n]{a_n} > 1$, so that $a_n > 1$ for $n \ge N$. The terms of the series do not converge to zero. The series diverges by the *n*th – term test for divergence.

3. $C = 1$. The series $\sum_{n=1}^{\infty}\dfrac{1}{n}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ show that the test is not conclusive when $C = 1$: the first series diverges and the second converges, but in both cases $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = 1$.

Example. Which of the following series converges and which diverges?

a) $\sum\limits_{n=1}^{\infty}\frac{1}{5^n}\left(\frac{n+1}{n}\right)^{n^2}$; **b)** $\sum\limits_{n=1}^{\infty}\frac{2^n}{n^2}$.

Solution.

The series $\sum\limits_{n=1}^{\infty}\frac{1}{5^n}\left(\frac{n+1}{n}\right)^{n^2}$ converges because

$$C = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{5^n}\left(\frac{n+1}{n}\right)^{n^2}} = \frac{1}{5}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \frac{1}{5}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{5} < 1.$$

But the series $\sum\limits_{n=1}^{\infty}\frac{2^n}{n^2}$ diverges because

$$C = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{2}{\left(\sqrt[n]{n}\right)^2} = \frac{2}{1} = 2 > 1.$$

§ 1.10. Alternating Series

Definition. A series in which the terms are alternately positive and negative is called an **alternating** series, that is

$$b_1 - b_2 + b_3 - b_4 + \ldots + (-1)^{n+1}b_n + \ldots \quad (b_n > 0) \tag{1.10.1}$$

Leibniz's theorem. The series (1.10.1) converges if the following conditions are satisfied:

1). $b_n \ge b_{n+1}$ for all *n*,

2). $\lim\limits_{n\to\infty} b_n = 0$.

Proof. If *n* is an even integer, say $n = 2m$, then the sum of the first *n* terms is

$S_{2m} = (b_1 - b_2) + (b_3 - b_4) + \ldots + (b_{2m-1} - b_{2m})$ or

$S_{2m} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots - (b_{2m-2} - b_{2m-1}) - b_{2m}$.

The first equality shows that S_{2m} is the sum of *m* nonnegative terms, since each term in parentheses is positive or zero. Hence $S_{2m+2} \ge S_{2m}$, and the sequence $\{S_{2m}\}$ is nondecreasing. The second equality shows that $S_{2m} \le b_1$. Since $\{S_{2m}\}$ is nondecreasing and bounded, from above, it has a limit, say

$$\lim_{m\to\infty} S_{2m} = L. \tag{1.10.2}$$

If *n* is an odd integer, say $n = 2m + 1$, then the sum of the first *n* terms is

$$S_{2m+1} = S_{2m} + b_{2m+1}.$$

Since $b_m \to 0$, $\lim\limits_{m\to\infty} b_{2m+1} = 0$ and as $m \to \infty$,

$$S_{2m+1} = S_{2m} + b_{2m+1} \to L + 0 = L. \qquad (1.10.3)$$

When we combine the results of (1.10.2) and (1.10.3), we get

$$\lim_{n\to\infty} S_n = L.$$

The theorem is proved.

Example. Test the alternating series:

$$\frac{1+\sqrt{2}}{2} - \frac{1+\sqrt{3}}{4} + \frac{1+\sqrt{4}}{6} - \frac{1+\sqrt{5}}{8} + \ldots$$

for convergence.

Solution. An alternating series is convergent if **(1)** the terms after a certain *n*th term, decrease numerically, i.e., $b_{n+1} \le b_n$, and **(2)** the general term approaches zero as n becomes infinite. Therefore, we determine the nth term of the given series. By discovering the law of formation, we find that $b_n = \dfrac{1+\sqrt{n+1}}{2n}$. Therefore, the preceding term is $\dfrac{1+\sqrt{n}}{2(n-1)}$. To satisfy the condition stated above, we must show that

$$\frac{1+\sqrt{n+1}}{2n} \le \frac{1+\sqrt{n}}{2(n-1)}.$$

Obtaining the common denominator for both these terms gives

$$\frac{\left(1+\sqrt{n+1}\right)(n-1)}{n(n-1)} \le \frac{\left(1+\sqrt{n}\right)n}{(n-1)n}.$$

Since the denominators are the same , to prove condition **(1)** we must show

$$\left(1+\sqrt{n+1}\right)(n-1) \le \left(1+\sqrt{n}\right)n.$$

This is obvious since subtracting 1 from n has a grater effect than adding 1 to $\sqrt{n}$. Since $b_{n+1} \le b_n$ we have the first condition for convergence.

Now we find the limit:

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{2(n-1)} = 0.$$

Since both conditions of Leibniz's theorem hold the given alternating series is convergent.

§ 1.11. Absolute Convergence

Definition. A series $\sum\limits_{n=1}^{\infty} a_n$ **converges absolutely** (is absolutely convergent) if the corresponding series of absolute values, $\sum\limits_{n=1}^{\infty} |a_n|$, converges.

Example. The alternating geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

converges absolutely because the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

is convergent.

Definition. A series that converges but does not converge absolutely converges **conditionally.**

The alternating harmonic series converges conditionally.

Theorem. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof. For each n:

$$-|a_n| \le a_n \le |a_n|,$$

so

$$0 \le a_n + |a_n| \le 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the comparison test, the nonnegative series $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ((a_n + |a_n|) - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

Example1. For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$, the corresponding series of absolute values is $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$, which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ because $|\sin n| \le 1$ for every n. The original series converges absolutely, therefore it converges.

Example 2. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1 + \sqrt{n+1}}{2n} \right)$ converges (see § 1.10), but it does not converge absolutely. In fact, corresponding series of absolute values is $\sum_{n=1}^{\infty} \left(\frac{1 + \sqrt{n+1}}{2n} \right)$, which diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

II. POWER SERIES

§ 2.1. The Radius and Interval of Convergence

Definition. A power series is the series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots \tag{2.1.1}$$

or

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_n(x - x_0)^n + \ldots \tag{2.1.2}$$

in which the **center** x_0 and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Definition. A point $x = a$ for which the number series

$$\sum_{n=0}^{\infty} c_n a^n = c_0 + c_1 a + c_2 a^2 + \ldots + c_n a^n + \ldots$$

is convergent is called a **point of convergence** of the series (2.1.1). The set of all points of convergence is the **domain of convergence** of this series.

Abel's theorem. If a series $\sum\limits_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$ converges for $x = x_0$ $(x_0 \neq 0)$, then it converges absolutely for all $|x| < |x_0|$. If the series diverges for $x = x_0$, then it diverges for all $|x| > |x_0|$.

Proof. Suppose the series $\sum\limits_{n=0}^{\infty} c_n x^n$ converges. Then $\lim\limits_{n \to \infty} c_n x^n = 0$.

Hence there is an integer *N* such that $|c_n x^n| < 1$ for all $n \geq N$. That is

$$|c_n| < \frac{1}{|x_o|^n} \quad \text{for} \quad n \geq N. \tag{2.1.3}$$

Now take any *x* such that $|x| < |x_0|$ and consider

$$|c_0| + |c_1 x| + \ldots + |c_{N-1} x^{N-1}| + |c_N x^N| + |c_{N+1} x^{N+1}| + \ldots$$

There is only a finite number of terms prior to $|c_N x^N|$ and their sum is finite. Starting with $|c_N x^N|$ and beyond, the terms are less than

$$\left|\frac{x}{x_0}\right|^N + \left|\frac{x}{x_0}\right|^{N+1} + \ldots \tag{2.1.4}$$

because of (2.1.3). But series (2.1.4) is a geometric series with ratio $q = \left|\dfrac{x}{x_0}\right|$, which is less than 1 since $|x| < |x_0|$. Hence the series (2.1.4) converges, so the original series (2.1.2) converges absolutely. This proves the first half of the theorem.

The second part of the theorem follows from the first. If the series diverges at $x = x_0$ and converges at a value x_1 with $|x_1| > |x_0|$, we conclude kipping in mind the first part of the theorem that the series converges absolutely at x_0. But the series can not converge absolutely and diverge at one and the same time. Hence, if it diverges at x_0, it diverges for all $|x| > |x_0|$.

Abel's theorem implies that all the points of convergence are not farther from the origin than any point of divergence. It is also clear that the points of convergence entirely cover the whole interval with the center at the origin. Thus we can say that for every power series possessing both points of convergence and points of divergence there is a positive number R such that for all x which absolute values are less than R that is $|x| < R,$ the series converges absolutely and for all x exceeding R in their absolute value, that is $|x| > R,$ it diverges.

Such R is called the **radius of convergence** of the power series. The interval $(-R, R)$ is referred to as the **interval of convergence.**

Now we calculate the radius of convergence of a power series. For this reason we consider the series

$$|c_0| + |c_1 x| + \left|c_2 x^2\right| + \ldots + \left|c_n x^n\right| + \ldots \tag{2.1.5}$$

composed of the absolute values of the terms of series (2.1.1). Suppose that the limit $\rho = \lim\limits_{n\to\infty} \dfrac{|c_{n+1}|}{|c_n|}$ exists and apply D'Alembert's test to the positive series (2.1.5)

$$D = \lim_{n\to\infty} \frac{|c_{n+1}| \cdot \left|x^{n+1}\right|}{|c_n| \cdot \left|x^n\right|} = |x| \cdot \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = |x| \cdot \rho .$$

The series (2.1.5) converges if $D < 1$ i.e. $|x| \cdot \rho < 1$ and $|x| < \dfrac{1}{\rho}$. Therefore the series (2.1.1) also converges for such x. If $D > 1$ that is $|x| > \dfrac{1}{\rho}$ than the series (2.1.5) diverges and also its general term does not tend to zero. But then the general term of the series (2.1.1) does not tend to zero and this series by the nth term test for divergence is divergent.

It should be noted that D'Alembert's test is inapplicable to the end points of the interval of convergence (the corresponding limit is equal to unity for these points), and therefore some other means should be used for testing the series for convergence.

So we have the following formula to define the radius of convergence of a power series

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}. \tag{2.1.6}$$

If we apply Cauchy's test we get

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|c_n|}} \tag{2.1.7}$$

Example 1. Find the domain of convergence of series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$$

Solution. First of all we determine the radius of convergence

$$R = \lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty}\frac{n+1}{n} = 1.$$

The series is absolutely convergent on the interval of convergence $(-1,1)$. For $x=-1$ we have the divergent series $-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}-\ldots$, and for $x=1$ we get the convergent series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$. So the domain of convergence of this series is $(-1;1]$.

Example 2. Find the domain of convergence of series

$$\frac{1}{2x} + \frac{2}{4x^2} + \frac{3}{8x^3} + \ldots + \frac{n}{2^n x^n} + \ldots$$

This is a power series in $\frac{1}{x}$.

Solution. Apply D'Alembert's test to obtain

$$D = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}x^{n+1}}\right|\left|\frac{2^n x^n}{n}\right| = \frac{1}{2|x|}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{2|x|}.$$

The series converges absolutely for $\frac{1}{2|x|}<1$ or $|x|>\frac{1}{2}$.

For $x=\frac{1}{2}$ the series becomes $1+2+3+4+\ldots$

and for $x=-\frac{1}{2}$ the series becomes $-1+2-3+4-\ldots$

Both of these series diverge. Thus the given series converges on the intervals $x<-\frac{1}{2}$ and $x>\frac{1}{2}$, and diverges on the closed interval $\left[-\frac{1}{2};\frac{1}{2}\right]$.

§ 2.2. Properties of Power Series

1. The sum of a power series is a continuous function in the interval of convergence of the series, i.e. the function

$$S(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots \qquad (-R<x<R)$$

is continuous.

2. A power series can be integrated termwise within its interval of convergence:

$$\int_0^x S(x)dx = \int_0^x c_0 dx + \int_0^x c_1 x dx + \int_0^x c_2 x^2 dx + \ldots + \int_0^x c_n x^n dx + \ldots =$$

$$= c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots + c_n \frac{x^{n+1}}{n+1} + \ldots, \ (-R < x < R).$$

The integrated series has the same radius of convergence as the original series.

3. A power series can be differentiated termwise in its interval of convergence:

$$S'(x) = c_1 + 2c_2 x + \ldots + nc_n x^{n-1} + \ldots, \ (-R < x < R).$$

The radius of convergence of the differentiated series is the same as that of the original series.

§ 2.3. Taylor's and Maclaurin's Series

Suppose the function $f(x)$ is infinitely differentiable in a neighborhood of a point x_0. Assume that it is represented as the sum of power series convergent in an interval containing the point x_0:

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_n(x - x_0)^n + \ldots \qquad (2.3.1)$$

Find coefficients $c_0, \ c_1, c_2, \ldots, c_n$. To do it let us set $x = x_0$ in equation (2.3.1). This yields

$$c_0 = f(x_0).$$

Now, let us differentiate series (2.3.1) and put $x = x_0$ into the differentiated series

$$f'(x) = c_1 + 2c_2(x - x_0) + \ldots + nc_n(x - x_0)^{n-1} + \ldots$$

This results in

$$c_1 = f'(x_0).$$

The repeated differentiation gives

$$f''(x) = 2c_2 + \ldots + n(n-1)c_n(x - x_0)^{n-2} + \ldots$$

whence, on substituting $x = x_0$ we obtain

$$c_2 = \frac{f''(x_0)}{2!}.$$

Proceeding in this way we have, after the *n*th differentiation

$$c_n = \frac{f^{(n)}(x_0)}{n!}. \qquad (2.3.2)$$

On substituting into formula (2.3.1) coefficients c_n defined by expression (2.3.2) we arrive at the series

$$f(x) = f(x_0) + f'(x_o)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 +$$

$$+ \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad (2.3.3)$$

which is called the **Taylor series** of the function $f(x)$.

If we set $x_0 = 0$ we get such a series:

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(x_0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \qquad (2.3.4)$$

which is called the **Maclaurin's series**.

The *n*th partial sum of series (2.3.3) is referred to as **Taylor's polynomial** of the *n*th order:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad (2.3.5)$$

Now we get the conditions under which the Taylor series (2.3.3) formed for a given function $f(x)$ converges in an interval and its sum exactly equal to $f(x)$.

Setting

$$f(x) = T_n(x) + R_n(x), \qquad (2.3.6)$$

$R_n(x)$ is called the **remainder** after *n* terms in Taylor's formula.

The convergence of the Taylor series to the function $f(x)$ at a point *x* is equivalent to the condition

$$\lim_{n \to \infty} T_n(x) = f(x) \qquad (2.3.7)$$

or which is the same to the relation

$$\lim_{n \to \infty} (f(x) - T_n(x)) = \lim_{n \to \infty} R_n(x) = 0 \qquad (2.3.8)$$

The value of $R_n(x)$ is exactly equal to the error appearing when the function $f(x)$ is replaced by Taylor's polynomial $T_n(x)$.

It is possible to represent the remainder $R_n(x)$ in the following Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \qquad (2.3.9)$$

where ξ is lying between x_0 and *x*.

Theorem. Let there exist a number $M > 0$ such that for all *x* in some interval containing the point x_0 and for all *n* the following inequality holds

$$\left| f^{(n)}(x) \right| \le M$$

then the function $f(x)$ can be expanded into Taylor's series in this interval.

Proof. By formula (2.3.9) we have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right| = \left| f^{(n+1)}(\xi) \right| \frac{|x - x_0|^{n+1}}{(n+1)!} \le M \frac{|x - x_0|^{n+1}}{(n+1)!}$$

Since the ratio $\frac{|x - x_0|^{n+1}}{(n+1)!}$ tends to zero for $n \to \infty$ it follows that

$$\lim_{n \to \infty} R_n(x) = 0$$

and we get

$$f(x) = f(x_0) + f'(x_o)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad (2.3.10)$$

§ 2.4. Expansion of Elementary Functions in Power Series

1. Let us expand the function $f(x) = e^x$ into Maclaurin's series.

Expressed in terms of x, the given function and its derivatives are:

$$f(x) = e^x, \quad f'(x) = e^x, \quad \ldots, f^{(n)}(x) = e^x, \ldots,$$

so

$$f(0) = 1, \quad f'(0) = 1, \quad \ldots, f^{(n)}(0) = 1, \ldots$$

Now we consider an interval $[-N, N]$ where N is an arbitrary fixed number. For all values of x of this interval we have $e^x < e^N = M$, therefore $\left|f^{(n)}(x)\right| \leq M$. According to the previous theorem the Maclaurin series generated by the function $f(x) = e^x$ converges and its sum is equal to e^x. So we have the expansion of the function $f(x) = e^x$ into Maclaurin's series as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \qquad (2.4.1)$$

The radius of convergence of this series is:

$$R = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim_{n \to \infty} (n+1) = \infty.$$

So the series (2.4.1) converges for all x.

2. Working in the same way we obtain such expansions:

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \ldots \quad (-\infty < x < \infty) \qquad (2.4.2)$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \ldots \quad (-\infty < x < \infty) \qquad (2.4.3)$

c) $(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \ldots +$

$$+ \frac{m(m-1)(m-2)\ldots(m-n+1)x^n}{n!} + \ldots \quad (-1 < x < 1) \qquad (2.4.4)$$

The series (2.4.4) is called the **binomial series**.

3. Let us expand the functions $f(x) = \ln(1+x)$ and $f(x) = \arctan x$ into Maclaurin's series.

Setting in the formula (2.4.4) $m=-1$ we have

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+\ldots \quad (-1<x<1) \qquad (2.4.5)$$

Integrating this series term by term we get

$$\int_0^x \frac{dx}{1+x}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

or

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots (-1<x<1) \qquad (2.4.6)$$

If we put $x=1$ we obtain

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}\frac{1}{n}+\ldots \qquad (2.4.7)$$

The expansion of the function $f(x)=\arctan x$ is found completely analogously. To this end we substitute x^2 for x in series (2.4.5) and integrate from 0 to x $(|x|<1)$ to receive

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots (-1<x<1) \qquad (2.4.8)$$

This series remains convergent at the end points of the interval of convergence, and therefore expansion (2.4.8) is also valid for $x=\pm 1$.

Substituting the value $x=1$ into this series we get the representation of the number π in the form of the numerical series:

$$\arctan 1=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+ \qquad (2.4.9)$$

Example 2.4.1. Obtain the Maclaurin expansion for $f(x)=e^{-x^2}$.
Solution. Replace x by $(-x^2)$ in the expansion for e^x (2.4.1);

$$e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots+(-1)^{n-1}\frac{x^{2n-2}}{(n-1)!}+\ldots (-\infty<x<\infty).$$

Example 2.4.2. Let us estimate the number of terms needed to compute the number e to within 0.00001.

Solution. Using the expansion (2.4.1) and setting $x=1$ by means of estimation (2.3.9) we have:

$$R_n<\frac{3}{(n+1)!}<10^{-5}.$$

It is easy to check that this inequality holds for $n=8$:

$$\frac{3}{9!}=\frac{3}{362880}<10^{-5}, \text{ so we obtain}$$

$$e\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{8!}\approx 2.71828.$$

Example 2.4.3. Evaluate $\int_0^1 \frac{\sin x}{x}dx$.

Solution. The difficulty here is that $\int \frac{\sin x}{x}dx$ can not be expressed in elementary functions. However,

$$\int_0^1 \frac{\sin x}{x}dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)dx = \left(x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + ...\right)\Bigg|_0^1 =$$

$$= 0.946083.$$

The error in using only four terms is less than or equal to $\frac{1}{9 \cdot 9!} = 0.0000003.$

Example 2.4.4. Estimate $\int_0^1 \sin x^2 dx$ with an error less than 0.001.

Solution. From the series for $\sin x$ (2.4.2) we have

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - ... \left(-\infty < x < \infty\right)$$

Therefore

$$\int_0^1 \sin x^2 dx = \left(\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - ...\right)\Bigg|_0^1 =$$

$$= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - ...$$

The series alternates, and we find by experiment that

$$\frac{1}{15 \cdot 7!} \approx 0.00076$$

is the first term to be numerically less than 0.001. The sum of preceding two terms gives

$$\int_0^1 \sin x^2 dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms we could estimate

$$\int_0^1 \sin x^2 dx \approx 0.310268$$

with an error of less than 10^{-6}, and with only one term beyond that we have.

$$\int_0^1 \sin x^2 dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303$$

with an error of less than 10^{-9}.

Example 2.4.5. In the theory of probability an important role is played by the function

$$\Phi(x)=\sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{x^2}{2}}\,dx$$

called **Laplace's function** or the **probability integral**. This integral cannot be computed in the finite form since $\int e^{-\frac{x^2}{2}}\,dx$ is inexpressible in terms of elementary functions. Expanding the integrand into a power series gives

$$e^{-\frac{x^2}{2}}=1-\frac{x^2}{2\cdot 1!}+\frac{x^4}{2^2 2!}-\frac{x^6}{2^3 3!}+\ldots\quad(-\infty<x<\infty).$$

The term by term integration now yields the representation of the function $\Phi(x)$ in the form of the infinite series:

$$\Phi(x)=\sqrt{\frac{2}{\pi}}\left(x-\frac{x^3}{3\cdot 2\cdot 1!}+\frac{x^5}{5\cdot 2^2 2!}-\frac{x^7}{7\cdot 2^3 3!}+\ldots\right)$$

convergent for all values of *x*.

III. Fourier Series

§ 3.1. Orthogonal Systems of Functions

Suppose that the functions $y = f(x)$ and $y = g(x)$ are given and continuous in the closed interval $[a,b]$.

Definition. The integral of the product $f(x)\cdot g(x)$ over the closed interval $[a,b]$ is called the **scalar product** of these functions and denoted by

$$(f,g) = \int_a^b f(x)\cdot g(x)dx \tag{3.1.1}$$

The scalar product (f,g) satisfies the following conditions:

1. $(f,g) = (g,f)$,

2. $(\lambda f,g) = \lambda(f,g)$,

3. $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$,

4. $(f,f) \geq 0$.

Definition. The functions $y = f(x)$ and $y = g(x)$ are said to be **orthogonal** in $[a,b]$ if their scalar product is equal to zero, that is:

$$(f,g) = \int_a^b f(x)\cdot g(x)dx = 0$$

Example. Check the orthogonality of the functions $y = x$ and $y = 3x - 2$ in the interval $[0,1]$.

Solution.

$$(x, 3x-2) = \int_0^1 x(3x-2)dx = \left(x^3 - x^2\right)\Big|_0^1 = 0.$$

Definition. The **norm** of the function $f(x)$ is

$$\|f\| = \sqrt{\int_a^b f^2(x)dx} \tag{3.1.2}$$

Definition. A system of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ is said to be orthogonal in the interval $[a,b]$ if the scalar product of any two different functions of this system is equal to zero:

$(\varphi_m, \varphi_n) = 0$ for $m \neq n$.

Theorem. System of functions

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos xn, \sin nx, \ldots$

is the orthogonal system of functions in the closed interval $[-\pi, \pi]$. This system is called the **trigonometric system** of functions.

Proof. Let us denote $C_n = \cos nx$, $\text{n} = 0,1,2,\ldots$, $S_n = \sin nx$, $n = 1,2,\ldots$ We should prove that:

1. $(C_0, C_n) = 0$, if $n \neq 0$,
2. $(C_0, S_n) = 0$,
3. $(C_m, S_n) = 0$,
4. $(C_m, C_n) = 0$, if $n \neq m$,
5. $(S_m, S_n) = 0$, if $n \neq m$,
6. $(C_0, C_0) = 2\pi$,
7. $(C_n, C_n) = \pi$,
8. $(S_n, S_n) = \pi$.

Now we prove the statements 3, 6, and 7.

3. $(C_m, S_n) = \int\limits_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = \frac{1}{2} \int\limits_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2} \int\limits_{-\pi}^{\pi} \sin(n-m)x dx = 0$,

since definite integrals in symmetric limits from odd functions are equal to zero.

6. $(C_0, C_0) = \int\limits_{-\pi}^{\pi} dx = 2\pi.$

7. $(C_n, C_n) = \int\limits_{-\pi}^{\pi} \cos^2 nx dx = \frac{1}{2} \int\limits_{-\pi}^{\pi} (1 + \cos 2nx) dx = \pi + \int\limits_{-\pi}^{\pi} \cos 2nx dx = \pi + 0 = \pi$.

The others can be proved in the same way.

§ 3.2. Fourier Series

Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ be the orthogonal system of functions in the interval $[a, b]$. Taking the sequence of numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ we form the series:

$$\alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \alpha_n \varphi_n(x) + \ldots \tag{3.2.1}$$

Suppose this series converges uniformly in the interval $[a, b]$ and its sum equals $f(x)$:

$$f(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \alpha_n \varphi_n(x) + \ldots \tag{3.2.2}$$

Let us find the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ To do this we multiply relation (3.2.2) by $\varphi_n(x)$ and integrate with respect to *x* from *a* to *b*:

$$\int\limits_a^b f(x)\varphi_n(x)dx = \alpha_1 \int\limits_a^b \varphi_1(x)\varphi_n(x)dx + \alpha_2 \int\limits_a^b \varphi_2(x)\varphi_n(x)dx + \ldots +$$
$$+ \alpha_1 \int\limits_a^b \varphi_n^2(x)dx + \ldots,$$

or

$$(f, \varphi_n) = \alpha_1(\varphi_1, \varphi_n) + \alpha_2(\varphi_2, \varphi_n) + \alpha_n(\varphi_n, \varphi_n) + \ldots \tag{3.2.3}$$

Because of the orthogonality of the system of functions $\{\varphi_n(x)\}$, where $n = 1,2,\ldots$ we have $(\varphi_m, \varphi_n) = 0$ for $m \neq n$.

Thus we get

$$\alpha_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)} \tag{3.2.4}$$

Suppose that the function $y = f(x)$ is integrable on the interval $[a, b]$.

Take any orthogonal system of functions $\{\varphi_n(x)\}$ where $n = 1,2,...$ and form the series

$$\alpha_1\,\varphi_1(x) + \alpha_2\,\varphi_2(x) + \alpha_n\,\varphi_n(x) + \ldots \tag{3.2.5}$$

where the coefficients $\alpha_n\ (n = 1,2,...)$ are defined by formula (3.2.4).

The series (3.2.5) is called the **generalized Fourier's series** of the function $y = f(x)$.

We can say that the function $f(x)$ generates its generalized Fourier's series

$$f(x) \sim \alpha_1\,\varphi_1(x) + \alpha_2\,\varphi_2(x) + \alpha_n\,\varphi_n(x) + \ldots \tag{3.2.6}$$

Coefficients $\alpha_n\ (n = 1,2,...)$ are called the **generalized the Fourier coefficients.**

Now let us take the trigonometric system of functions:

$$1, \cos x, \sin x,\ \cos 2x, \sin 2x, ..., \cos xn, \sin nx, ...$$

In this case we get

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{3.2.7}$$

where $\dfrac{a_0}{2} = \dfrac{(f, C_0)}{(C_0, C_0)} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx \Rightarrow$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{3.2.8}$$

$$a_n = \frac{(f, C_n)}{(C_n, C_n)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad b_n = \frac{(f, S_n)}{(S_n, S_n)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \tag{3.2.9}$$

The series (3.2.7) is called the **Fourier series** and the coefficients (3.2.8), (3.2.9) are called the **Fourier coefficients**.

We cannot assert that the Fourier series (3.2.7) of an arbitrary function $f(x)$ is convergent and its sum (in case of the convergence) is equal to the function $f(x)$.

Now we state without proof the key theorem on expanding a given function $f(x)$ into the Fourier series.

The Dirichlet Theorem. Let a function $f(x)$ together with its derivative $f'(x)$ be continuous or have finite number of points of discontinuity of the first kind in the closed interval $[-\pi, \pi]$. Then the Fourier series converges at any point of this interval and its sum $S(x)$ is defined by the formula:

1. $S(x) = f(x)$ if x is the interior point of the interval $[-\pi, \pi]$ at which the function is continuous.

2. $S(x)=\dfrac{f(x-0)+f(x+0)}{2}$ if x is the interior point of discontinuity of the first kind in the interval $[-\pi,\pi]$.

3. $S(-\pi)=S(\pi)=\dfrac{f(-\pi+0)+f(\pi-0)}{2}$.

The trigonometric series

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{3.2.10}$$

can be represented as a sum of simple harmonics only involving the sines. To this end we combine the summands $\cos nx$ and $\sin nx$ (n = 1, 2,…) with the same frequencies and transform the combination

$$a_n \cos nx + b_n \sin nx$$

by putting $a_n = A_n \sin \varphi_n$ and $b_n = A_n \cos \varphi_n$. This results in

$$a_n \cos nx + b_n \sin nx = A_n \sin(nx+\varphi_n).$$

In this equality

$A_n = \sqrt{a_n^2+b_n^2}$ is the amplitude of vibration, and

$\varphi_n = \arctan\dfrac{a_n}{b_n}$ is the initial phase of the nth harmonic,

$n=\omega_n$ is a circular frequency,

$T=\dfrac{2\pi}{\omega_n}$ is the period.

Series (3.2.10) then takes the form

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} A_n \sin(nx+\varphi_n) \tag{3.2.11}$$

Example. Expand the function $f(x)=\begin{cases}-1, & -\pi\le x<0\\ 1, & 0\le x\le\pi\end{cases}$ into the Fourier series.

Solution. This function is odd and discontinuous at the point $x=0$. We extend this function periodically with the period 2π for all points of $0x$-axis and find the Fourier coefficients:

$a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx dx = 0,\ n=0,1,2,...,$ since the integrand is odd.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi}1\cdot\sin nx dx=\frac{-2}{\pi n}(\cos\pi n-1)=\frac{2}{\pi n}\left(1-(-1)^n\right),$$

since $\cos\pi n=(-1)^n$.

Finely we obtain:

$$b_n = \begin{cases} 0, & \text{if } n = 2m, \\ \dfrac{4}{\pi n}, & \text{if } n = 2m+1. \end{cases}$$

The given function $f(x)$ satisfies the conditions of the Dirichlet theorem and we get

$f(x) = \dfrac{4}{\pi} \sum\limits_{m=0}^{\infty} \dfrac{\sin(2m+1)x}{2m+1}$, for all x, at which the function is continuous.

§ 3.3. Expanding Functions with Arbitrary Period

Let us consider the problem of expanding into Fourier's series function $f(x)$ defined in an interval $[-\ell, \ell]$. If in the interval $[-\ell, \ell]$, the function $f(x)$ satisfies the conditions of Dirichlet's theorem its expansion can be obtain by changing variable according to the formula $t = \dfrac{\pi}{\ell} x$. When the variable x runs through the interval $[-\ell, \ell]$ the variable t runs through the interval $[-\pi, \pi]$. The expansion of the function $f\left(\dfrac{\ell}{\pi} t\right)$ into Fourier's series has the form:

$$f\left(\frac{\ell}{\pi} t\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{3.3.1}$$

On returning back to the old variable, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}, \tag{3.3.2}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \cos ntdt = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \; (n = 0, 1, 2, 3, \ldots), \tag{3.3.3}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \sin ntdt = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx, \; (n = 1, 2, 3, \ldots). \tag{3.3.4}$$

The sum of Fourier's series is a periodic function of x with the period $T = 2\ell$. The frequencies of the harmonics of series (3.3.2) are $\omega_n = \dfrac{\pi n}{\ell}$, $(n = 1, 2, 3, \ldots)$.

§ 3.4. Expanding Even and Odd Functions into the Fourier Series

Assume that a function $f(x)$ is even. Then the functions $f(x)\sin nx$, $n = 1, 2, \ldots$ are odd and all the coefficients b_n are zero. Consequently the Fourier series of an even function consists only of cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{3.4.1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \text{c} \qquad (3.4.2)$$

If the function $f(x)$ is an odd function, then the functions $f(x)\cos nx$ are odd, and all the coefficients a_n are zero. In this case the Fourier series consists only of sines:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad (3.4.3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx, (n = 1,2,...), \qquad (3.4.4)$$

since $f(x)\sin nx$ are even functions.

Example. Expand the function $y = |\,x\,|$ given in the interval $(-\pi, \pi]$ into Fourier's series.

Solution. For the function $y = |\,x\,|$ is an even function then the Fourier coefficients are defined by formulas (3.4.2) :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\frac{x^2}{2}\Bigg|_0^{\pi} = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \quad \cos nx dx = \quad \frac{2}{\pi}\left(\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\Bigg|_0^{\pi} =$$

$$\begin{bmatrix} 1 & \frac{1}{n}\sin nx \\ 0 & -\frac{1}{n^2}\cos nx \end{bmatrix} \quad = \frac{2}{\pi}\left(\frac{\pi}{n}\sin \pi n + \frac{1}{n^2}\cos \pi n - \frac{1}{n^2}\cos 0\right) =$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right) = -\frac{4}{\pi(2k-1)^2},$$

$b_n = 0.$

The function $y = |\,x\,|$ satisfies the conditions of the Dirichlet theorem and we have:

$$|\,x\,| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ... + + \frac{\cos(2k-1)x}{(2k-1)^2} + ...\right).$$

§ 3.5. Fourier's Series in the Complex Form

Consider a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \qquad (3.5.1)$$

On replacing $\cos\frac{n\pi x}{\ell}$ and $\sin\frac{n\pi x}{\ell}$ by their expressions given by Euler's formulas

$$\cos\alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}, \qquad \sin\alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}$$

we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{\frac{in\pi x}{\ell}} + \frac{a_n - ib_n}{2} e^{-\frac{in\pi x}{\ell}} \tag{3.5.2}$$

Let us denote

$$c_0 = \frac{a_0}{2},\ c_n = \frac{a_n - ib_n}{2},\ c_{-n} = \frac{a_n + ib_n}{2}$$

and write

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{\ell}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{\ell}}$$

or

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{\ell}}, \tag{3.5.3}$$

where

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\left(\cos\frac{n\pi x}{\ell} - i\sin\frac{n\pi x}{\ell}\right)dx =$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{\frac{-in\pi x}{\ell}} dx,$$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{\frac{in\pi x}{\ell}} dx.$$

Therefore for all $n = 0, \pm 1, \pm 2, \ldots$ we obtain

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{\frac{-in\pi x}{\ell}} dx \tag{3.5.4}$$

The expression $e^{\frac{in\pi x}{\ell}}$ is called a **harmonic**;

Numbers $\omega_n = \frac{n\pi}{\ell}$ are called **spectral** or **wave numbers**.

A set of wave numbers is called a **spectrum** of $f(x)$.

The coefficients c_n are referred to as **complex amplitudes** of a function $f(x)$.

§ 3.6. Fourier's Integral

Suppose the function $f(x)$ satisfies the following conditions:

1. The function $f(x)$ is bounded and absolutely integrable on the interval $(-\infty,\infty)$, that is the improper integral $\int_{-\infty}^{\infty}|f(x)|dx$ exists and is finite.
2. The function $f(x)$ satisfies the Dirichlet conditions in any finite interval $[-\ell,\ell]$.

Take any number ℓ and write the expansion of the function $f(x)$ into the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\omega_n x+b_n\sin\omega_n x, \tag{3.6.1}$$

where

$$\omega_n=\frac{n\pi}{\ell},\ a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\cos\omega_n t dt,\ b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(t)\sin\omega_n t dt \tag{3.6.2}$$

The substitution of a_n and b_n into the Fourier's series (3.6.1) gives

$$\begin{aligned}f(x)&=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(t)dt+\frac{1}{\ell}\sum_{n=1}^{\infty}\int_{-\ell}^{\ell}f(t)(\cos\omega_n t\cos\omega_n x+sin\omega_n t\sin\omega_n x)dt=\\&=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(t)dt+\frac{1}{\ell}\sum_{n=1}^{\infty}\int_{-\ell}^{\ell}f(t)\cos\omega_n(t-x)dt\end{aligned} \tag{3.6.3}$$

Let us introduce the notation $\Delta\omega_n=\omega_{n+1}-\omega_n=\frac{\pi}{\ell}$ and substitute it into the expression (3.6.3):

$$f(x)=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\ell}^{\ell}f(t)\cos\omega_n(t-x)dt\right)\Delta\omega_n \tag{3.6.4}$$

If $\ell\to\infty$ then the first term tends to zero, the integral $\int_{-\ell}^{\ell}f(t)\cos\omega_n(t-x)dt$ tends to the improper integral $\int_{-\infty}^{\infty}f(t)\cos\omega_n(t-x)dt$ and the sum of the formula (3.6.4) tends to the integral with respect to ω

$$\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\omega(t-x)dt\right)d\omega.$$

Finally we obtain

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\omega(t-x)dt\right)d\omega. \tag{3.6.5}$$

This formula is called the **Fourier integral formula** and the integral is referred to as the **Fourier integral**.

For a periodic function with period 2ℓ we have its Fourier's expansion consisting of an infinite number of harmonics with frequencies $\omega_n = \frac{n\pi}{\ell}$ differing from each other by the constant $\Delta\omega = \frac{\pi}{\ell}$. Every such harmonic has a **discrete spectrum**. For a nonperiodic function defined in the interval $(-\infty, +\infty)$ and representable as Fourier's integral we can say that this function has a **continuous spectrum**. In this case the frequencies of the harmonics vary continuously.

§ 3.7. Fourier Integral for Even and Odd Functions

Since $\cos\omega(t-x) = \cos\omega t\cos\omega x + \sin\omega t\sin\omega x$ the formula (3.6.5) can be rewritten in the form

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\omega t dt\right)\cos\omega x d\omega + \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\omega t dt\right)\sin\omega x d\omega.$$

Suppose the $f(x)$ is an even function. Then the function $f(t)\sin\omega$ t is odd ant the integral $\int_{-\infty}^{+\infty} f(t)\sin\omega t d$t vanishes. Besides, in this case

$$\int_{-\infty}^{+\infty} f(t)\cos\omega t d\text{t} = 2\int_0^{+\infty} f(t)\cos\omega t d\text{t}.$$

Consequently, for even functions the Fourier integral takes the form

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(t)\cos\omega t dt\right)\cos\omega x d\omega \tag{3.7.1}$$

Similarly, if $f(x)$ is an odd function we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(t)\sin\omega t dt\right)\sin\omega x d\omega \tag{3.7.2}$$

Example. Let us represent as the Fourier integral the function

$$f(x) = \begin{cases} 1 \text{ for } 0 \le x \le 1, \\ 0 \text{ for } x > 1 \end{cases}$$

by extending it as an even function and as an odd function.

Solution. In the former case we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\omega x d\omega\int_0^1 \cos\omega t dt = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega} d\omega.$$

In the latter case (the odd extension) we obtain:

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\omega x d\omega\int_0^1 \sin\omega t dt = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega x(1-\cos\omega)}{\omega} d\omega.$$

§ 3.8. Fourier Integral in Complex Form

Consider the Fourier integral formula

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\omega(x-t)dt\right)d\omega.$$

Using Euler's formulas we have

$$\cos\omega(x-t)=\frac{e^{i\omega(x-t)}+e^{-i\omega(x-t)}}{2}$$

and

$$f(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\left(e^{i\omega(x-t)}+e^{-i\omega(x-t)}\right)dt\right)d\omega=$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)e^{i\omega(x-t)}dt\right)d\omega+\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\left(e^{-i\omega(x-t)}\right)dt\right)d\omega.$$

On substituting $z=-\omega$ in the second integral we verify that this integral equals the first integral. Therefore we arrive at the formula:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)e^{i\omega(x-t)}dt\right)d\omega \tag{3.8.1}$$

This formula is called the expansion of the function $f(x)$ into the Fourier integral in complex form.

Example. Let us represent as the Fourier integral in complex form the function

$$f(x)=\begin{cases}1 \text{ for} \mid x\mid\le\ell,\\ 0 \text{ for} \mid x\mid>\ell.\end{cases}$$

Solution. We have $\int_{-\infty}^{\infty}f(t)e^{i\omega(x-t)}dt=\int_{-\ell}^{\ell}e^{i\omega(x-t)}dt=\frac{2e^{i\omega x}}{\omega}\sin\omega\ell$.

So

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega\ell}{\omega}e^{i\omega x}d\omega.$$

§ 3.9. Fourier Transformation

Let us write the Fourier integral in the real form:

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\omega(t-x)dt\right)d\omega=\frac{1}{\pi}\int_{0}^{\infty}\left[\left(\int_{-\infty}^{\infty}f(t)\cos\omega t\,dt\right)\cos\omega x+\right.$$

$$\left.+\left(\int_{-\infty}^{\infty}f(t)\sin\omega t\,dt\right)\sin\omega x\right]d\omega. \tag{3.9.1}$$

We introduce the notation

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt, \qquad B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt,$$

(3.9.2)

We can rewrite the expression (3.9.1) in the following form:

$$f(x)=\int_{0}^{\infty}(A(\omega)\cos\omega x + B(\omega)\sin\omega x)d\omega \qquad (3.9.3)$$

The relations (3.9.2) and (3.9.3) are called **the Fourier transformations in the real form**.

If the function $f(x)$ is an even function then we get

$$f_c(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t dt \qquad (3.9.4)$$

and

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_c(\omega)\cos\omega x d\omega \qquad (3.9.5)$$

If $f(x)$ is an odd function then

$$f_s(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t dt \qquad (3.9.6)$$

and

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_s(\omega)\sin\omega x d\omega \qquad (3.9.7)$$

The functions $f_c(\omega)$ and $f_s(\omega)$ are called, respectively, the **Fourier cosine transform** and the **Fourier sine transform** of the function $f(x)$.

The function $f(x)$ in formulas (3.9.5) and (3.9.7) is called **Fourier's inverse transform** in the both cases.

Let us write a complex Fourier integral in the following form

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt\right)e^{-i\omega x}d\omega \qquad (3.9.8)$$

On setting

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

we obtain from formula (3.9.8)

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \qquad (3.9.9)$$

The function $F(\omega)$ is called the **Fourier transform** of the function $f(x)$ and is a complex function of the real argument ω. The original function $f(x)$ is the **Fourier inverse transform** of the function $F(\omega)$.

Example. Let us find the Fourier's transform of the function $f(x)=\begin{cases} \mathrm{e}^{-ax} & \text{for } x>0, \\ 0 & \text{for } x<0 \end{cases}$ $(a>0)$.

Solution. We have

$$F(\omega)=\int_0^{\infty} e^{-ax}\cdot e^{-i\omega x}dx=-\left.\frac{e^{ix(a+i\omega)}}{a+i\omega}\right|_0^{\infty}=\frac{1}{a+i\omega}.$$

§ 3.10. Properties of the Fourier Transform

1. The Fourier transform $F(\omega)$ of an absolutely integrable function $f(x)$ is bounded and continuous function, which tends to zero as $|\,\omega\,|\to\infty$.

2. The Fourier transform is a linear operator, that is

$$F\{\lambda_1 f_1+\lambda_2 f_2\}=\lambda_1 F\{f_1\}+\lambda_2 F\{f_2\}. \qquad (3.10.1)$$

3. The Fourier transform of derivatives:

$$F\{f'(t)\}=(-i\omega)F\{f(t)\} \qquad (3.10.2)$$

4. If a function $f(x)$ is absolutely integrable on the interval $(-\infty,+\infty)$ then

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(\omega)|^2d\omega \qquad (3.10.3)$$

The integral $\int_{-\infty}^{\infty}|f(x)|^2dx$ is called a **normalized energy *E*** scattered by the current $f(x)$ passing through the resistance in one Ohm.

The function $|F(\omega)|^2$ is called the **spectrum** of density of energy and shows the relative contribution of different frequencies into the total energy.

5. Given two functions $f(x)$ and $g(x)$, both absolutely integrable on $(-\infty,+\infty)$. Let S denote the set of x for which the improper integral

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)g(x-t)dt \qquad (3.10.4)$$

converges. This integral defines a function $h(x)$ on S called the **convolution of $f(t)$ and $g(t)$.** We also write $h=f*g$ to denote this function.

The integral (3.10.4) is encountered in signal processing and other fields.

The Fourier transform of the convolution product of two functions is equal to the product of their Fourier's transforms, i.e.

$$F\{f * g\} = F\{f\} \cdot F\{g\} \tag{3.10.5}$$

Formula (3.10.5) is the basis and essence of all linear signal processing. Its meaning is that a linear system acts on a signal as a frequency filter.

All these five properties are given without proof.

§ 3.11. Spectral Properties of the Fourier Series and Integral

Sets of coefficients a_n and b_n $(n = 1,2,...)$ of the expansion of a periodic function $f(t)$ into the Fourier series are called frequency spectrums of this function. If a period of a function is equal to 2ℓ, then $a_n = a_n(n\Delta\omega)$, $b_n = b_n(n\Delta\omega)$, where $\Delta\omega = \dfrac{\pi}{\ell}$ is the frequency of the firs harmonic. Hence spectrums are the functions depending on the number of harmonic as an independent variable.

The quantity $A_n = \sqrt{a_n^2 + b_n^2}$ is termed the **amplitude spectrum** of the function $f(t)$.

Let us write the Fourier series in a complex form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Delta\omega t}, \tag{3.11.1}$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(t) e^{-in\Delta\omega t} dt, \quad n = 0, \pm 1, \pm 2, ...$$

The expression

$$S(n\Delta\omega) = 2c_n \ell = \int_{-\ell}^{\ell} f(t) e^{-in\Delta\omega t} dt$$

is referred to as the spectrum function or the spectrum density of the Fourier series.

The complex function $S(n\Delta\omega)$ is defined by the module and the phase.

The module of the spectrum function is called the **amplitude spectrum**, i.e.

$$\rho(n\Delta\omega) = |S(n\Delta\omega)| = \left| \int_{-\ell}^{\ell} f(t) e^{-in\Delta\omega t} dt \right|, \tag{3.11.2}$$

and $\Phi(n\Delta\omega) = -\arg S(n\Delta\omega)$ is the **phase spectrum.**

The expression

$$S(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{3.11.3}$$

is called the **spectrum function** or the **spectrum density** of the Fourier integral.

$\rho(\omega) = |S(\omega)|$ is termed the **amplitude spectrum** and

$\Phi(\omega) = -\arg S(\omega)$ is the **phase spectrum**.

Example 3.11.1. Find the spectral characteristics of the periodic square impulse

$$f(t) = \begin{cases} E, & 0 < t < \tau, \\ 0, & \tau < \mathrm{t} < \mathrm{T} \end{cases} \;;\; f(t+T) = f(t).$$

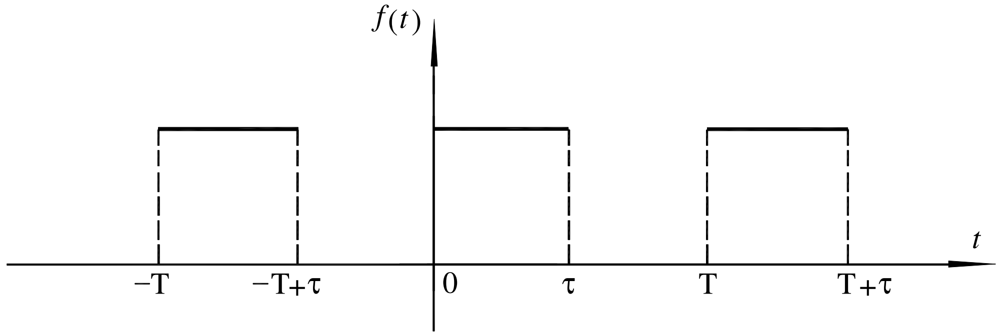

Fig. 3.11.1

Solution. Let us find the spectrum function:

$$S(n\Delta\omega) = 2c_n\ell = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2in\Delta\omega t} dt = i\frac{E}{2n\Delta\omega}(\cos 2n\Delta\omega\tau - 1 - i\sin 2n\Delta\omega\,\tau) =$$

$$= -i\frac{E}{2n\Delta\omega}\left(2\sin^2 n\Delta\omega\,\tau + i\sin 2n\Delta\omega\,\tau\right).$$

The amplitude spectrum:

$$\rho(n\Delta\omega) = |S(n\Delta\omega)| = \frac{E}{|n\Delta\omega|}|\sin n\Delta\omega\,\tau|.$$

The phase spectrum:

$$\Phi(n\Delta\omega) = -\arg S(n\Delta\omega) = \arctan\frac{2\sin^2 n\Delta\omega\,\tau}{\sin 2n\Delta\omega\,\tau} =$$

$$= \arctan\frac{2\sin^2 n\Delta\omega\,\tau}{2\sin n\Delta\omega\,\tau\cdot\cos n\Delta\omega\,\tau} = \arctan(\tan n\Delta\omega\,\tau) = n\Delta\omega\,\tau.$$

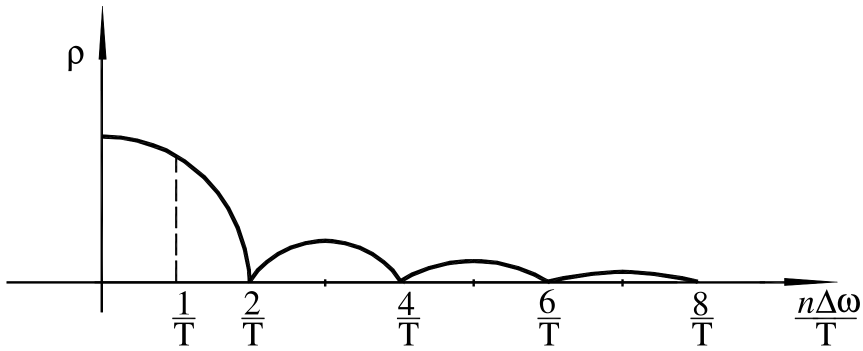

Fig. 3.11.2

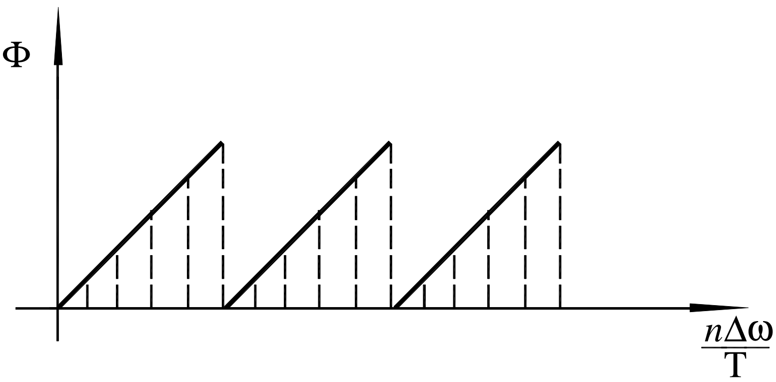

Fig.3.11.3

Example 3.11.2 . Find the spectral characteristics of the trapezoidal impulse

$$f(t) = \begin{cases} t + 2.5, & -2.5 \le t \le 1.5, \\ 1, & -1.5 \le \mathrm{t} \le 1.5, \\ 2.5\text{-}t, & 1.5 \le t \le 2.5, \\ 0, & |\,\mathrm{t}\,| > 2.5. \end{cases}$$

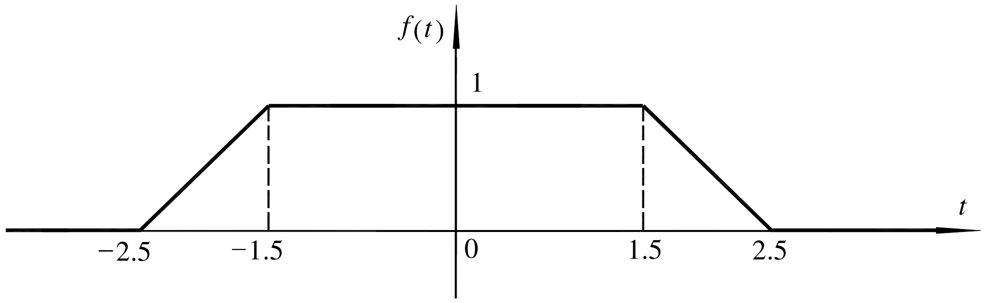

Fig. 3.11.4

Solution. The spectrum function of the impulse $f(t)$ is

$$S(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt=\int_{-2.5}^{+2.5} f(t)e^{-i\omega t}\,dt=\int_{-2.5}^{+2.5} f(t)(\cos\omega t-i\sin\omega t)dt=$$

$$=2\int_{0}^{+2.5} f(t)\cos\omega t\,dt=\frac{4}{\omega^2}\sin 2\omega\cdot\sin\frac{\omega}{2}.$$

The graph of the amplitude spectrum of the impulse $f(t)$

$$|S(\omega)|=\frac{4}{\omega^2}\left|\sin 2\omega\cdot\sin\frac{\omega}{2}\right| \text{ is}$$

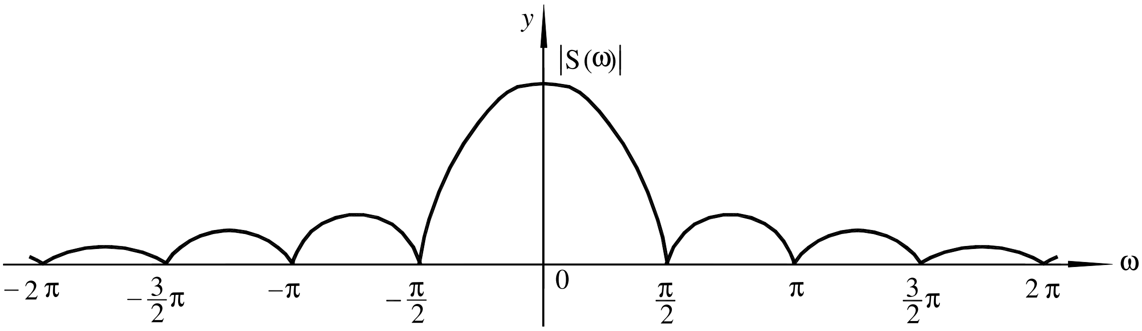

Fig. 3.11.5

IY. Miscellaneous Problems

4.1. Investigate the convergence or divergence of the following series:

1) $a_n = \dfrac{1}{n^2+1}$ **2)** $a_n = \dfrac{2}{\sqrt{n+3}}$ **3)** $a_n = \dfrac{n+2}{\sqrt{3n}}$

4) $a_n = \sin\dfrac{\pi}{3^n}$ **5)** $a_n = \dfrac{n!}{2^n}$ **6)** $a_n = \dfrac{(2n-1)!!}{3^n \cdot n!}$

7) $a_n = \dfrac{n^2}{(n+1)!}$ **8)** $a_n = \dfrac{3}{\sqrt{n}\cdot e^{\sqrt{n}}}$ **9)** $a_n = \dfrac{n(n+1)}{3^n}$

10) $a_n = \left(\dfrac{n}{2n+1}\right)^n$ **11)** $a_n = 3^n\left(\dfrac{n}{n+1}\right)^{n^2}$ **12)** $a_n = \arctan\dfrac{n}{n+1}$

13) $a_n = \dfrac{\ln n}{n}$ **14)** $a_n = \dfrac{1+\cos\left(\pi - \dfrac{1}{n}\right)}{n^2}$ **15)** $a_n = \dfrac{n^2}{5n^2+1}$

16) $a_n = \sqrt{\left(\dfrac{n-1}{n}\right)^n}$ **17)** $a_n = \sqrt{\left(\dfrac{2n-1}{n+1}\right)^n}$ **18)** $a_n = \arctan\sqrt{n}$

4.2. Which of the following series are absolutely convergent, which are conditionally convergent, and which are divergent?

1) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{(2n-1)^3}$ **2)** $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{10n+1}$ **3)** $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}n^2}{2n^2+3}$

4) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}2n^3}{n!}$ **5)** $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n\cdot\ln^3 n}$ **6)** $\sum\limits_{n=1}^{\infty}\left(\dfrac{n+3}{n}\right)^n$

4.3. Prove that:

1) $\lim\limits_{n\to\infty}\dfrac{5^n}{n!} = 0$ **2)** $\lim\limits_{n\to\infty}\dfrac{n^n}{(2n)!} = 0$

3) $\lim\limits_{n\to\infty}\dfrac{n!}{(2n)!!} = 0$ **4)** $\lim\limits_{n\to\infty}\dfrac{(n!)^n}{n^{n^2}} = 0$

4.3. Find the domain of convergence of series:

1) $\sum\limits_{n=1}^{\infty}\dfrac{(x-1)^n}{4^n}$ **2)** $\sum\limits_{n=1}^{\infty}\dfrac{n^5}{x^n}$ **3)** $\sum\limits_{n=1}^{\infty}\dfrac{(x-2)^{n-1}n^2}{n\cdot 9^n}$

4) $\sum_{n=1}^{\infty} \frac{x^{2n} n!}{n^5}$ **5)** $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)^2 \cdot 2^{2n}}$ **6)** $\sum_{n=1}^{\infty} \frac{(x-1)^n}{4n}$

4.4. Obtain the Maclaurin expansion for following functions:

1) $y = \cos^2 x$ **2)** $y = e^{2x} + 2e^{-x}$ **3)** $y = \frac{1}{(1+x)^2}$

4) $y = \frac{1}{\sqrt[3]{1+x}}$ **5)** $y = \arcsin x$ **6)** $y = \ln(1 - x^2)$

4.5. Estimate the given integrals with an error less than 0.001.

1) $\int_0^1 \arctan\left(\frac{x^2}{2}\right) dx$ **2)** $\int_0^1 \frac{\ell n(1+x)}{x} dx$ **3)** $\int_0^{1/2} \frac{1-\cos x}{x^2} dx$

4.6. Estimate the given expressions with an error less than 0.001.

1) $\cos 9^0$ **2)** $\sqrt[3]{68}$ **3)** $\ell n 1.04$ **4.** $\frac{1}{\sqrt[4]{e}}$

4.7. Expand the given functions into Fourier's series.

1. $f(x) = \begin{cases} -2x, & -\pi \le x < 0 \\ 3x, & 0 \le x \le \pi \end{cases}$ **2)** $f(x) = 4 - x, \text{ if } x \in (0, 4)$

3) $f(x) = x^2, \text{ if } x \in (-\pi, \pi)$ **4.** $f(x) = 4 - |x|, \text{ if } x \in (-4, 4)$

Навчальний посібник

В. Н. Гавдзинский, ***Л. Н. Коробова***

SERIES
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